Article:

Beresnevich, Victor orcid.org/0000-0002-1811-9697, Ganguly, Arijit, Ghosh, Anish et al. (1 more author) (2020) Inhomogeneous dual Diophantine approximation on affine subspaces. International Mathematics Research Notices. 3582–3613. ISSN 1687-0247

https://doi.org/10.1093/imrn/rny124

V. Beresnevich *et al.* (2018) "Inhomogeneous Dual Diophantine Approximation on Affine Subspaces,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–32
Advance Access publication on Month XX, 2018
doi:10.1093/imrn/rny124

Inhomogeneous Dual Diophantine Approximation on Affine Subspaces

Victor Beresnevich[2], Arijit Ganguly[1,3], Anish Ghosh[1,*], and Sanju Velani[2]

[1]School of Mathematics, Tata Institute of Fundamental Research, Mumbai, 400005, India, [2]Department of Mathematics, University of York, Heslington, York YO10 5DD, UK, and [3]School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel

[*]*Correspondence to be sent to: e-mail: ghosh@math.tifr.res.in*

We prove the convergence and divergence cases of an inhomogeneous Khintchine–Groshev-type theorem for dual approximation restricted to affine subspaces in $\mathbb{R}^n$. The divergence results are proved in the more general context of Hausdorff measures.

1 Introduction

Throughout the paper, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function and $\mathcal{W}_n(\psi)$ is the set of $\mathbf{x} \in \mathbb{R}^n$ for which there exist infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that

$$|a_0 + \mathbf{x} \cdot \mathbf{a}| < \psi(\|\mathbf{a}\|^n) \tag{1.1}$$

for some $a_0 \in \mathbb{Z}$. Here and throughout, $\| \ \|$ denotes the supremum norm of a vector and the dot stands for the standard inner product of vectors. For obvious reasons, the set $\mathcal{W}_n(\psi)$ is often referred to as the (dual) set of "ψ-approximable" vectors in $\mathbb{R}^n$. The fundamental Khintchine–Groshev theorem [22, 23] in the metric theory of Diophantine approximation provides an elegant characterisation of the n-dimensional Lebesgue measure of $\mathcal{W}_n(\psi)$ in terms of the convergence/divergence properties of a "volume sum" associated with the *approximating* function ψ. We reinforce the fact that ψ will always be assumed to be nonincreasing.

Received January 13, 2018; Revised May 12, 2018; Accepted May 15, 2018

Theorem 1.1. (Khintchine–Groshev) Let ψ be an approximating function. Then

$$\left|\mathcal{W}_n(\psi)\right| = \begin{cases} 0 & \text{if } \sum_{k=1}^{\infty} \psi(k) < \infty \\ \text{full} & \text{if } \sum_{k=1}^{\infty} \psi(k) = \infty. \end{cases} \tag{1.2}$$

We will use $|\quad|$ to denote the absolute value of a real number as well as the Lebesgue measure of a measurable subset X of $\mathbb{R}^n$; the context will make the use clear.

Remarks.

(1) By Dirichlet's theorem, $\mathcal{W}_n(\psi) = \mathbb{R}^n$ when $\psi(k) = k^{-1}$.

(2) A point $\mathbf{x} \in \mathbb{R}^n$ is called *very well approximable* (VWA) if there exists $\varepsilon > 0$ such that $\mathbf{x} \in \mathcal{W}_n(\psi_\varepsilon)$, where

$$\psi_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+ : k \to \psi_\varepsilon(k) := k^{-(1+\varepsilon)}.$$

Thus, the essence of the definition of VWA points is that for these points, the "Dirichlet exponent" can be improved beyond the trivial. Note that in view of Theorem 1.1, we have that $|\mathcal{W}_n(\psi_\varepsilon)| = 0$ for any $\varepsilon > 0$. In other words, almost every point $\mathbf{x} \in \mathbb{R}^n$ is not VWA.

(3) The more general Hausdorff measure version of the Khintchine–Groshev theorem has been established in [14]. For a general background to the classical theory of metric Diophantine approximation, we refer the reader to the survey-type articles [7, 9].

We now consider the setting of inhomogeneous Diophantine approximation. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function and, given ψ, we let $\mathcal{W}_n^\theta(\psi)$ be the set of $\mathbf{x} \in \mathbb{R}^n$ for which there exist infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that

$$\left|a_0 + \mathbf{x} \cdot \mathbf{a} + \theta(\mathbf{x})\right| < \psi(\|\mathbf{a}\|^n) \tag{1.3}$$

for some $a_0 \in \mathbb{Z}$. The set $\mathcal{W}_n^\theta(\psi)$ is often referred to as the (dual) set of "(ψ, θ)-inhomogeneously approximable" vectors in $\mathbb{R}^n$. The following inhomogeneous version of Theorem 1.1 is established in [2]. We denote by C^n the set of n-times continuously differentiable functions.

Theorem 1.2. Let ψ be an approximating function and $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta \in C^2$. Then

$$\left|\mathcal{W}_n^\theta(\psi)\right| = \begin{cases} 0 & \text{if } \sum_{k=1}^{\infty} \psi(k) < \infty \\ \text{full} & \text{if } \sum_{k=1}^{\infty} \psi(k) = \infty. \end{cases} \tag{1.4}$$

We remark that the choice of $\theta =$ constant is the setting of traditional inhomogeneous Diophantine approximation and in that case the above result was well known, see for example [12]. For the more general Hausdorff measure version of Theorem 1.2 within the traditional setting, see [6] and [7, §12.1].

In this paper, we consider the theory of Diophantine approximation on manifolds, specifically inhomogeneous approximation on affine subspaces. The subject of metric Diophantine approximation on manifolds studies the conditions under which a smooth submanifold of $\mathbb{R}^n$ inherits Diophantine properties of $\mathbb{R}^n$, which are generic for Lebesgue measure. Examples include the resolution of the famous Baker–Sprindžuk conjecture [27] due to Kleinbock and Margulis [26] using homogeneous dynamics on the space of unimodular lattices. Their result implies that almost every point on a nondegenerate submanifold $\mathcal{M}$ of $\mathbb{R}^n$ is not VWA; that is,

$$\left|\mathcal{W}_n(\psi_\varepsilon) \cap \mathcal{M}\right|_{\mathcal{M}} = 0 \quad \forall \ \varepsilon > 0. \tag{1.5}$$

Here and elsewhere $| \ . \ |_{\mathcal{M}}$ denotes the induced Lebesgue measure on $\mathcal{M}$. It is worth mentioning that any manifold $\mathcal{M}$ of $\mathbb{R}^n$ satisfying (1.5) is called extremal and that Kleinbock and Margulis proved the stronger "multiplicative" extremal statement for nondegenerate manifolds. Essentially, nondegenerate manifolds are smooth manifolds of $\mathbb{R}^n$ that are sufficiently curved so as to deviate from any hyperplane, see [3, 26] for a formal definition.

The convergence case of the Khintchine–Groshev theorem was shown to hold for nondegenerate submanifolds of $\mathbb{R}^n$ in [11] and independently in [3]. Indeed, in [11], Bernik, Kleinbock, and Margulis established the stronger "multiplicative" version. The complementary divergence case was subsequently proved in [10] and, as a result, we have the following complete analogue of the Khintchine–Groshev theorem for nondegenerate manifolds.

Theorem 1.3. Let $\mathcal{M}$ be a nondegenerate submanifold of $\mathbb{R}^n$, and let ψ be an approximating function. Then

$$\left|\mathcal{W}_n(\psi) \cap \mathcal{M}\right|_{\mathcal{M}} = \begin{cases} 0 & \text{if } \sum_{k=1}^{\infty} \psi(k) < \infty \\ \text{full} & \text{if } \sum_{k=1}^{\infty} \psi(k) = \infty. \end{cases} \tag{1.6}$$

We note that the convergence case of the above theorem implies the extremal statement (1.5) for nondegenerate manifolds. The reader is also referred to [1, 13] for recent developments concerning Diophantine approximation on nondegenerate manifolds. In this paper, we are concerned with affine subspaces, which are the main examples of manifolds that are not nondegenerate. Theorem 1.3 above is therefore not applicable to them. Nevertheless, the analogue of the Baker–Sprindžuk conjecture for affine subspaces was studied by Kleinbock in [24] (see also [25]), and the Khintchine–Groshev theorem in the series of papers [8, 15, 16, 17, 19]. We refer the reader to the recent survey [20] for further details on this subject. The key goal of this paper is to develop the analogous inhomogeneous theory for affine subspaces.

We now briefly describe the current state of the inhomogeneous theory of Diophantine approximation on manifolds. In [4, 5], the authors discovered a transference principle that allowed them to establish the inhomogeneous version of the Baker–Sprindžuk conjecture for nondegenerate manifolds from the original homogeneous statement. Indeed, the inhomogeneous "multiplicative" extremal statement established in [4] implies that for any nondegenerate submanifold $\mathcal{M}$ of $\mathbb{R}^n$ and $\theta =$ constant,

$$\left|\mathcal{W}_n^{\theta}(\psi_\varepsilon) \cap \mathcal{M}\right|_{\mathcal{M}} = 0 \quad \forall \ \varepsilon > 0. \tag{1.7}$$

It is worth mentioning that in [21], it has been shown that the homogeneous to inhomogeneous transference principle of [4] is flexible enough to be used for arbitrary Diophantine exponents, not just the critical or "extremal" one. As demonstrated in [21], this naturally extends the scope of potential applications of the original transference principle. Beyond extremal statements such as (1.7), the complete inhomogeneous version of the Khintchine–Groshev theorem, both convergence and divergence cases, for nondegenerate manifolds is established in [2]. In other words, with mild conditions imposed on the "inhomogeneous" function θ, the statement of Theorem 1.3 is shown to be valid with $\mathcal{W}_n(\psi)$ replaced by $\mathcal{W}_n^{\theta}(\psi)$. In fact, in the divergence case, for any $\theta \in C^2$ the more general Hausdorff measure version is established. As is to be expected, the convergence case of the Khintchine–Groshev theorem established in [2] implies the inhomogeneous extremal statement (1.7) for nondegenerate manifolds. To the best of our knowledge, unlike in the homogeneous setting, an inhomogeneous theory of Diophantine approximation on affine subspaces is yet to be developed. As already alluded to above, the purpose of this work is to address this imbalance by establishing an inhomogeneous version of the Khintchine–Groshev theorem for affine subspaces of $\mathbb{R}^n$. As a consequence, we obtain the inhomogeneous extremal statement (1.7) for affine subspaces. Indeed, our results go some way towards developing a coherent inhomogeneous theory for degenerate manifolds as outlined in [2, §1.4].

In the study of Diophantine approximation on affine subspaces, one needs to assume some condition on the slope of the affine subspace in order to ensure that the affine subspace inherits generic Diophantine properties from its ambient Euclidean space. We will now introduce certain Diophantine exponents of matrices that play a key role in this regard. Indeed, we need these exponents in order to even state our main convergence theorem.

1.1 Diophantine exponents of matrices

Throughout $\mathscr{H}$ will be an open subset of a d-dimensional affine subspace of $\mathbb{R}^n$. By making a change of variables, if necessary, we can assume without loss of generality that $\mathscr{H}$ is of the form

$$\left\{(\mathbf{x}, \mathbf{x}A' + \mathbf{a}_0) : \mathbf{x} \in U\right\}, \tag{1.8}$$

where $\mathbf{a}_0 \in \mathbb{R}^{n-d}$ and $A' \in \mathrm{Mat}_{d\times n-d}(\mathbb{R})$ and U is an open subset of $\mathbb{R}^d$. On setting

$$A := \begin{pmatrix} \mathbf{a}_0 \\ A' \end{pmatrix},$$

we can rewrite this parametrisation as

$$\mathbf{x} \mapsto (\mathbf{x}, \tilde{\mathbf{x}}A), \quad \text{where } \tilde{\mathbf{x}} := (1, \mathbf{x}). \tag{1.9}$$

Given a column $\boldsymbol{\theta} \in \mathbb{R}^{d+1}$ and a matrix $A \in \mathrm{Mat}_{d+1\times n-d}(\mathbb{R})$, the inhomogeneous Diophantine exponent $\omega(A;\boldsymbol{\theta})$ of $(A;\boldsymbol{\theta})$ is defined to be the supremum of $v > 0$ for which there are infinitely many $\mathbf{a}' \in \mathbb{Z}^{n-d} \setminus \{\mathbf{0}\}$ such that

$$\left\|A\mathbf{a}' + \mathbf{a}'' + \boldsymbol{\theta}\right\| < \|\mathbf{a}'\|^{-v} \tag{1.10}$$

for some $\mathbf{a}'' \in \mathbb{Z}^{d+1}$. In the case $\boldsymbol{\theta} = \mathbf{0}$, $\omega(A) := \omega(A;\mathbf{0})$ is the usual (homogeneous) Diophantine approximation exponent of the matrix A. It is well known that $(n-d)/(d+1) \le \omega(A) \le \infty$ for all $A \in \mathrm{Mat}_{d+1\times n-d}(\mathbb{R})$ and that $\omega(A) = (n-d)/(d+1)$ for Lebesgue almost every A.

We now introduce the higher Diophantine exponents of A as defined by Kleinbock in [25]. For $A \in \mathrm{Mat}_{d+1\times n-d}(\mathbb{R})$, we set

$$R_A := (\mathrm{Id}_{d+1}\ A) \ \in\ \mathrm{Mat}_{d+1\times n+1}(\mathbb{R}), \tag{1.11}$$

where Id_{d+1} denotes the $(d+1)\times(d+1)$ identity matrix. Let $\mathbf{e}_0,\dots,\mathbf{e}_n$ denote the standard basis of $\mathbb{R}^{n+1}$ and set

$$W_{i\to j} := \mathrm{span}\{\mathbf{e}_i,\dots,\mathbf{e}_j\}, \tag{1.12}$$

where $i,j \in \{0,\cdots,n\}$ with $i \le j$, be the linear subspace of $\mathbb{R}^{n+1}$ spanned by vectors $\mathbf{e}_i,\dots,\mathbf{e}_j$. Clearly, $W_{0\to n} = \mathbb{R}^{n+1}$. Now let $\mathbf{w} \in \bigwedge^j(W_{0\to n})$ represent a discrete subgroup Γ of $\mathbb{Z}^{n+1}$ of rank j, that is, $\mathbf{w}$ is the wedge product of vectors from any given basis of Γ. Define the map

$$\mathbf{c} : \bigwedge^j(W_{0\to n}) \to \left(\bigwedge^{j-1}(W_{1\to n})\right)^{n+1}$$

by setting

$$\mathbf{c}(\mathbf{w})_i := \sum_{\substack{J\subset\{1,\dots,n\}\\ \#J=j-1}} \left\langle \mathbf{e}_i \wedge \mathbf{e}_J, \mathbf{w}\right\rangle \mathbf{e}_J \tag{1.13}$$

for $0 \le i \le n$, and let $\pi_\bullet$ denote the projection $\bigwedge(W_{0\to n}) \to \bigwedge(W_{d+1\to n})$. For each $j = 1,\dots,n-d$, define

$$\omega_j(A) := \sup\left\{ v : \begin{array}{c} \exists\, \mathbf{w} \in \bigwedge^j(\mathbb{Z}^{n+1}) \text{ with arbitrarily large } \|\pi_\bullet(\mathbf{w})\| \\ \text{such that } \|R_A\, \mathbf{c}(\mathbf{w})\| < \|\pi_\bullet(\mathbf{w})\|^{-\frac{v+1-j}{j}} \end{array} \right\}. \tag{1.14}$$

It is shown in Lemma 5.3 of [25] that $\omega_1(A) = \omega(A)$. We shall see in the next section that the Diophantine exponents $\omega_j(A)$ play a key role in the convergence case of the Khintchine–Groshev theorem for affine subspaces.

1.2 Our main theorems

As before $\mathscr{H}$ will denote an open subset of a d-dimensional affine subspace of $\mathbb{R}^n$ parametrised as in (1.9). Then, given ψ and θ, the object of study is the set $\mathcal{W}_n^\theta(\psi) \cap \mathscr{H}$; that is, the set of "$(\psi,\theta)$-inhomogeneously approximable" vectors on $\mathscr{H}$ (the reason for considering an open subset of an affine subspace rather than the whole subspace is to allow inhomogeneous functions θ that may not necessarily be defined on the whole subspace, for example, $\theta(\mathbf{x}) = \sqrt{1-(x_1^2+\cdots+x_d^2)}$).Our first result establishes the convergence case of the inhomogeneous Khintchine–Groshev theorem for affine subspaces.

Theorem 1.4. Let $\mathscr{H}$ be an open subset of an affine subspace of $\mathbb{R}^n$ of dimension d given by (1.9), and suppose that

$$\omega_j(A) < n \text{ for every } j = 1, \ldots, n-d. \tag{1.15}$$

Let ψ be an approximating function and $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathscr{H}}$ is analytic. Further in the case $\theta|_{\mathscr{H}}$ is a linear function so that

$$\hat{\theta}(\mathbf{x}) := \theta(\mathbf{x}, \tilde{\mathbf{x}} A) = \tilde{\mathbf{x}}\boldsymbol{\theta} = \theta_0 + \theta_1 x_1 + \cdots + \theta_d x_d \tag{1.16}$$

for some column $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_d)^t$, assume that

$$\omega(A; \boldsymbol{\theta}) < n. \tag{1.17}$$

Then,

$$\left|\mathcal{W}_n^\theta(\psi) \cap \mathscr{H}\right|_{\mathscr{H}} = 0 \tag{1.18}$$

whenever

$$\sum_{k=1}^{\infty} \psi(k) \; < \; \infty. \tag{1.19}$$

Recall that $|\ |_{\mathscr{H}}$ denotes the induced Lebesgue measure on $\mathscr{H}$. Clearly, the above convergence theorem implies the inhomogeneous extremal statement (1.7) for any affine subspace satisfying (1.15) and any analytic $\hat{\theta}$ that additionally satisfies (1.17) in the case it is linear.

Remarks.

(1) In the case $\theta|_{\mathscr{H}}$ is linear, condition (1.17) on the exponent of $(A; \boldsymbol{\theta})$ is optimal. Indeed, suppose that

$$\|A\mathbf{a}' + \mathbf{a}'' + \boldsymbol{\theta}\| < \|\mathbf{a}'\|^{-n} \log^{-3} \|\mathbf{a}'\| \tag{1.20}$$

holds for infinitely many $\mathbf{a}' \in \mathbb{Z}^{n-d} \setminus \{\mathbf{0}\}$ and some $\mathbf{a}'' \in \mathbb{Z}^{d+1}$, but

$$\|A\mathbf{a}' + \mathbf{a}'' + \boldsymbol{\theta}\| < \|\mathbf{a}'\|^{-n} \log^{-4} \|\mathbf{a}'\| \tag{1.21}$$

holds only for finitely many $\mathbf{a}' \in \mathbb{Z}^{n-d} \setminus \{\mathbf{0}\}$ and $\mathbf{a}'' \in \mathbb{Z}^{d+1}$. Clearly, in this case $\omega(A; \boldsymbol{\theta}) = n$. The existence of such pairs $(A; \boldsymbol{\theta})$ can be proved by using the inhomogeneous version of Jarnik's theorem for systems of linear forms [9, Theorem 19]. Assuming $\mathbf{a}'$ and $\mathbf{a}''$ satisfy (1.20), write $\mathbf{a}'$ as $(a_{d+1},\ldots,a_n)^t$ and $\mathbf{a}''$ as $(a_0,\ldots,a_d)^t$. Then, on identifying $\mathbf{a}$ with $(a_1,\ldots,a_n)^t$ one readily verifies that

$$\hat{\theta}(\mathbf{x})+a_0+(\mathbf{x},\tilde{\mathbf{x}}A)\mathbf{a}=\tilde{\mathbf{x}}(A\mathbf{a}'+\mathbf{a}''+\boldsymbol{\theta}) \tag{1.22}$$

and that $\|\mathbf{a}\| \ll \|\mathbf{a}'\|$. Therefore,

$$\left|\hat{\theta}(\mathbf{x})+a_0+(\mathbf{x},\tilde{\mathbf{x}}A)\mathbf{a}\right| \ll \|A\mathbf{a}'+\mathbf{a}''+\boldsymbol{\theta}\| \ll \|\mathbf{a}\|^{-n}\log^{-3}\|\mathbf{a}\|,$$

where the implied constant, which depends on $\mathbf{x}$, can be chosen uniformly for $\mathbf{x}$ in a compact set. Take $\psi(h)=h^{-1}(\log h)^{-2}$. Then, clearly for every $\mathbf{x}$ in such a compact set, the inequality

$$\left|\hat{\theta}(\mathbf{x})+a_0+(\mathbf{x},\tilde{\mathbf{x}}A)\mathbf{a}\right| < \psi(\|\mathbf{a}\|^n)$$

holds for infinitely many $\mathbf{a}\in\mathbb{Z}^{n-d}\setminus\{\mathbf{0}\}$ and some $a_0\in\mathbb{Z}$. In this case

$$\mathcal{W}_n^{\theta}(\psi)\cap\mathscr{H}=\mathscr{H}$$

despite (1.19). An obvious modification of the above argument shows that, given an approximating function ψ satisfying (1.19), in the case $\theta|_{\mathscr{H}}$ is linear, (1.18) necessarily implies the existence of $c>0$ such that

$$\|A\mathbf{a}'+\mathbf{a}''+\boldsymbol{\theta}\| \ge c\,\psi\left(\|(\mathbf{a}',\mathbf{a}'')\|^n\right) \qquad \text{for all } (\mathbf{a}',\mathbf{a}'')\in\mathbb{Z}^{n-d}\setminus\{\mathbf{0}\}\times\mathbb{Z}^{d+1}.$$

How close this is to being a sufficient condition remains an interesting question that is open even in the homogeneous case.

(2) We note that in case $\boldsymbol{\theta}=\mathbf{0}$, since $\omega(A):=\omega(A;\mathbf{0})$ the inhomogeneous Diophantine condition (1.17) does not add extra hypotheses in the homogeneous case.

For the divergence counterpart to Theorem 1.4, we shall prove the following more general statement in terms Hausdorff measures. Throughout, $\mathcal{H}^s(X)$ will denote the s-dimensional Hausdorff measure of a subset X of $\mathbb{R}^n$ and $\dim X$ the Hausdorff dimension, where $s>0$ is a real number.

Theorem 1.5. Let $\mathscr{H}$ be an open subset of an affine subspace of $\mathbb{R}^n$ of dimension d and let $s > d-1$. Let ψ be an approximating function and $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathscr{H}} \in C^2$. Suppose that (1.15) holds and that

$$\sum_{k=1}^{\infty} k^{\frac{d-s}{n}} \psi(k)^{s+1-d} = \infty. \tag{1.23}$$

Then

$$\mathcal{H}^s\left(\mathcal{W}_n^\theta(\psi) \cap \mathscr{H}\right) = \mathcal{H}^s(\mathscr{H}). \tag{1.24}$$

Given an approximating function ψ, the lower order at infinity τ_ψ of $1/\psi$ is defined by

$$\tau_\psi := \liminf_{t\to\infty} \frac{-\log \psi(t)}{\log t} \tag{1.25}$$

and indicates the growth of $1/\psi$ "near" infinity. Now observe that the divergent sum condition (1.23) is satisfied whenever

$$s < d - 1 + (n+1)/(n\tau_\psi + 1).$$

Therefore, it follows from the definition of Hausdorff dimension that

$$\dim\left(\mathcal{W}_n^\theta(\psi) \cap \mathscr{H}\right) \geq s \quad \text{if} \quad \mathcal{H}^s\left(\mathcal{W}_n^\theta(\psi) \cap \mathscr{H}\right) > 0$$

and that $\mathcal{H}^s(\mathscr{H}) > 0$ if $s \leq \dim \mathscr{H} = d$ and $\mathscr{H} \neq \emptyset$. We therefore obtain the following dimension statement concerning the set $\mathcal{W}_n^\theta(\psi) \cap \mathscr{H}$.

Corollary 1.1. Let $\mathscr{H}$ be a nonempty open subset of an affine subspace of $\mathbb{R}^n$ of dimension d. Let ψ be an approximating function and $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathscr{H}} \in C^2$. Suppose that (1.15) holds and that $1 \leq \tau_\psi < \infty$. Then

$$\dim\left(\mathcal{W}_n^\theta(\psi) \cap \mathscr{H}\right) \geq d - 1 + \frac{n+1}{n\tau_\psi + 1}. \tag{1.26}$$

Remarks.

(1) To the best of our knowledge, the above findings constitute the first known results in the context of inhomogeneous Diophantine approximation on affine subspaces. In fact, Theorem 1.5 is new even for Lebesgue measure (i.e., when $s = d$) and in many cases for the homogeneous setting (i.e.,

when $\theta \equiv 0$). The only previously known cases in the homogeneous setting were the following:

(a) the case of lines passing through the origin, which was treated in [8], and

(b) the case of affine hyperplanes ($d = n - 1$), which was treated in [18].

(2) In the case $d = n - 1$ condition (1.15) represents a single inequality imposed on the main Diophantine exponent $\omega(A)$ of A.

(3) In the case $s = d$ the sum within (1.23) matches the one within (1.19). Thus, Theorem 1.5 naturally complements the statement of Theorem 1.4.

(4) In [2], the smoothness condition imposed on the inhomogeneous function θ is weaker than what we have assumed to establish the convergence statement of Theorem 1.4. In short we have imposed the stronger analyticity condition to deal with a technical problem involving (C, α)-good functions (see below for the definition). It is plausible that this condition can be relaxed and brought at par with that imposed in [2].

(5) The homogeneous results in [11] and the inhomogeneous convergence results in [2] are proved in the context of more general multivariable approximating functions. This setting includes the case of "multiplicative" Diophantine approximation. Both our main theorems should hold for nondegenerate submanifolds of affine subspaces and our convergence theorem should, in addition, be true in the multivariable setting. We plan to return to this extension in a separate work.

2 The Gradient Division

In this section, we prepare the groundwork to prove Theorem 1.4, the "convergence case". Let U be the same as in (1.8) and as before define $\hat{\theta} : U \longrightarrow \mathbb{R}$ by setting

$$\hat{\theta}(\mathbf{x}) := \theta(\mathbf{x}, \tilde{\mathbf{x}}A).$$

Clearly, $\hat{\theta}$ is an analytic function since $\theta|_{\mathscr{H}}$ is analytic. For $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, we define

$$\mathcal{L}(\mathbf{a}) := \left\{ \mathbf{x} \in U : \left| \hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a} \right| < \psi(\|\mathbf{a}\|^n) \text{ for some } a_0 \in \mathbb{Z} \right\}.$$

Observe that $\limsup \mathcal{L}(\mathbf{a})$ is the projection of $\mathcal{W}_n^{\theta}(\psi) \cap \mathscr{H}$ onto $\mathbb{R}^d$. Here and elsewhere, unless stated otherwise, $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and any unspecified limsup is taken over such $\mathbf{a}$.

Since the projection from $\mathscr{H}$ to $\mathbb{R}^d$ is bi-Lipschitz, Theorem 1.4 will follow on showing that $|\limsup \mathcal{L}(\mathbf{a})| = 0$. In fact, it is sufficient to show that for each $\mathbf{x} \in U$, we can choose an open ball B centred at $\mathbf{x}$ with $11B \subseteq U$ such that

$$\left|\limsup \mathcal{L}(\mathbf{a}, B)\right| = 0\,, \tag{2.1}$$

where $\mathcal{L}(\mathbf{a}, B) := \mathcal{L}(\mathbf{a}) \cap B$.

To prove the above measure zero statement, it is natural to consider separately the case when we have a "large derivative" and the case when we do not. More precisely, we split the set $\mathcal{L}(\mathbf{a}, B)$ into two subsets depending on the size of the quantity $\nabla(\hat{\theta}(\mathbf{x}) + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}) = \nabla(\hat{\theta}(\mathbf{x})) + [\mathrm{Id}_d\ A']\mathbf{a}^t$—here A' is as introduced at the start of Section 1.1 and as usual ∇ denotes the gradient operator. With this in mind, for any sufficiently small open ball B with $11B \subseteq U$, we define

$$\mathcal{L}_{small}(\mathbf{a}, B) = \left\{\mathbf{x} \in \mathcal{L}(\mathbf{a}, B)\ :\ \left\|\nabla\big(\hat{\theta}(\mathbf{x}) + (\mathbf{x}, \tilde{\mathbf{x}}A)\cdot\mathbf{a}\big)\right\| < \sqrt{n\mathrm{d}L\|\mathbf{a}\|}\right\}, \tag{2.2}$$

where

$$L := \max\left\{\sup_{|\beta|=2,\ \mathbf{x}\in 2B}\left\|\partial_\beta\hat{\theta}(\mathbf{x})\right\|, \frac{1}{4r^2}\right\} \tag{2.3}$$

and r is the radius of B. Here for a multi-index $\beta = (i_1, \ldots, i_d)$ of non-negative integers $|\beta| := i_1 + \cdots + i_d$ and ∂_β denotes the corresponding differentiation operator, that is, $\frac{\partial^{|\beta|}}{\partial x_1^{i_1} \ldots \partial x_d^{i_d}}$. Set $\mathcal{L}_{large}(\mathbf{a}, B) = \mathcal{L}(\mathbf{a}, B) \backslash \mathcal{L}_{small}(\mathbf{a}, B)$. We will prove that for any "appropriately chosen" $B \subseteq 11B \subseteq U$,

$$\left|\limsup \mathcal{L}_{large}(\mathbf{a}, B)\right| = 0 \tag{2.4}$$

and

$$\left|\limsup \mathcal{L}_{small}(\mathbf{a}, B)\right| = 0\,. \tag{2.5}$$

Clearly, on combining the measure zero statements (2.4) and (2.5) we obtain the desired measure zero statement (2.1).

3 Estimating the Measure of $\limsup \mathcal{L}_{large}(\mathbf{a}, B)$

In this section, we will establish (2.4) as a simple consequence of the following statement.

Proposition 3.1. [11, Lemma 2.2] Let $B \subseteq \mathbb{R}^d$ be a ball of radius r and $F \in C^2(2B)$, where $2B$ is the ball with the same centre as B and radius $2r$. Let

$$M^* := \sup_{|\beta|=2,\,\mathbf{x}\in 2B} \left\|\partial_\beta F(\mathbf{x})\right\| \tag{3.1}$$

and

$$M := \max\left\{M^*, \frac{1}{4r^2}\right\}. \tag{3.2}$$

Then for every $\delta' > 0$, the set of all $\mathbf{x} \in B$ such that $|p + F(\mathbf{x})| < \delta'$ for some $p \in \mathbb{Z}$ and

$$\left\|\nabla F(\mathbf{x})\right\| \geq \sqrt{\mathrm{d}M} \tag{3.3}$$

has measure at most $K_d\delta'|B|$, where $K_d > 0$ is a constant dependent only on d.

To prove (2.4) from Proposition 3.1, we start with any sufficiently small open ball B in U. We fix $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and take $F(\mathbf{x}) = ((\mathbf{x}, \tilde{\mathbf{x}}A), \hat{\theta}(\mathbf{x})) \cdot (\mathbf{a}, 1)$ for $\mathbf{x} \in 2B$ and $\delta' = \psi(\|\mathbf{a}\|^n)$. Clearly, $M = L$, where L is given by (2.3). Hence, by Proposition 3.1, we get that

$$\left|\mathcal{L}_{large}(\mathbf{a}, B)\right| \leq K_d\psi(\|\mathbf{a}\|^n)|B|,$$

and thus

$$\sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \left|\mathcal{L}_{large}(\mathbf{a}, B)\right| \leq K_d \sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \psi(\|\mathbf{a}\|^n)|B| \ll \sum_{h=1}^{\infty} h^{n-1}\psi(h^n) \ll \sum_{h=1}^{\infty}\psi(h).$$

Since the latter sum is convergent, on applying the Borel–Cantelli lemma we obtain (2.4), as desired. In the above, as well as elsewhere, $\ll$ means an inequality with an unspecified multiplicative constant.

In order to establish (2.5) we will use the "inhomogeneous transference principle" introduced in [4, Section 5], whose simplified version is recalled in the next section.

4 Inhomogeneous Transference Principle

Throughout this section, we shall let B be an open ball in $\mathbb{R}^d$ and ϱ be the d-dimensional Lebesgue measure restricted to B so that the closed ball $\overline{B}$ becomes the support of ϱ.

Consider two countable index sets $\mathcal{T}$, $\mathcal{A}$ and two maps $H : (t, \alpha, \eta) \mapsto H_t(\alpha, \eta)$ and $I : (t, \alpha, \eta) \mapsto I_t(\alpha, \eta)$ from $\mathcal{T} \times \mathcal{A} \times \mathbb{R}_+$ to the collection of all open sets in $\mathbb{R}^d$. Take a set

Φ of functions $\phi : \mathcal{T} \longrightarrow \mathbb{R}_+$. For each $\phi \in \Phi$, define

$$\Lambda_I(\phi) := \limsup_{t\in\mathcal{T}} \bigcup_{\alpha\in\mathcal{A}} I_t\big(\alpha, \phi(t)\big) \quad \text{and} \quad \Lambda_H(\phi) := \limsup_{t\in\mathcal{T}} \bigcup_{\alpha\in\mathcal{A}} H_t\big(\alpha, \phi(t)\big).$$

We now discuss the two main properties that enables one to transfer zero ϱ-measure statements for the "homogeneous" limsup sets $\Lambda_H(\phi)$ to the "inhomogeneous" limsup sets $\Lambda_I(\phi)$.

(1) **Intersection property**: The triplet (H, I, Φ) is said to satisfy the intersection property if for any $\phi \in \Phi$, there exists $\phi^* \in \Phi$ such that for all but finitely many $t \in \mathcal{T}$ and for all distinct $\alpha, \alpha' \in \mathcal{A}$, we have

$$I_t\big(\alpha, \phi(t)\big) \cap I_t\big(\alpha', \phi(t)\big) \subseteq \bigcup_{\alpha''\in\mathcal{A}} H_t\big(\alpha'', \phi^*(t)\big)\,. \tag{4.1}$$

(2) **Contracting property**: We say that ϱ is contracting with respect to (I, ϕ) if for any $\phi \in \Phi$, there exists $\phi^+ \in \Phi$, a sequence of positive numbers $\{k_t\}_{t\in\mathcal{T}}$ with $\sum_{t\in\mathcal{T}} k_t < \infty$ and for all but finitely many $t \in \mathcal{T}$ and all $\alpha \in \mathcal{A}$, a collection $\mathcal{C}_{t,\alpha}$ of balls $\mathfrak{B}$ centred in $\overline{B}$ satisfying the three conditions given below:

$$\overline{B} \cap I_t\big(\alpha, \phi(t)\big) \subseteq \bigcup_{\mathfrak{B}\in\mathcal{C}_{t,\alpha}} \mathfrak{B}\,, \tag{4.2}$$

$$\overline{B} \cap \bigcup_{\mathfrak{B}\in\mathcal{C}_{t,\alpha}} \mathfrak{B} \subseteq I_t\big(\alpha, \phi^+(t)\big), \tag{4.3}$$

and

$$\varrho\big(5\mathfrak{B} \cap I_t\big(\alpha, \phi(t)\big)\big) \leq k_t \varrho(5\mathfrak{B}). \tag{4.4}$$

The main transference for our purpose, which follows easily from [4, Theorem 5], can be stated as follows.

Theorem 4.1. If (H, I, Φ) satisfies the intersection property and ϱ is contracting with respect to (I, ϕ) then

$$\forall\, \phi \in \Phi,\ \varrho\big(\Lambda_H(\phi)\big) = 0 \qquad \Longrightarrow \qquad \forall\, \phi \in \Phi,\ \varrho\big(\Lambda_I(\phi)\big) = 0.$$

5 Proof of (2.5) from Theorem 4.1

Let $\mathcal{T} := \mathbb{Z}_+, \mathcal{A} = (\mathbb{Z}^n \setminus \{\mathbf{0}\}) \times \mathbb{Z}$. For $t \in \mathcal{T}, \alpha := (\mathbf{a}, a_0) \in \mathcal{A}$ and $\eta \in \mathbb{R}_+$, we set

$$I_t(\alpha,\eta) := \left\{ \mathbf{x} \in U : \left| \begin{array}{l} \left|\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\right| < \frac{\eta}{2^{nt}} \\ \left\|\nabla(\hat{\theta}(\mathbf{x}) + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a})\right\| < \sqrt{n d L} \times \eta \times 2^{t/2} \\ 2^t \le \|\mathbf{a}\| < 2^{t+1} \end{array} \right. \right\} \tag{5.1}$$

and

$$H_t(\alpha,\eta) := \left\{ \mathbf{x} \in U : \left| \begin{array}{l} \left|a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\right| < \frac{2\eta}{2^{nt}} \\ \left\|\nabla(\mathbf{x}, \tilde{\mathbf{x}}A) \cdot \mathbf{a}\right\| < 2\sqrt{n d L} \times \eta \times 2^{t/2} \\ \|\mathbf{a}\| < 2^{t+2} \end{array} \right. \right\}. \tag{5.2}$$

Given $\delta \in \mathbb{R}$, define

$$\phi_\delta : \mathcal{T} \longrightarrow \mathbb{R}_+ : t \to \phi_\delta(t) := 2^{\delta t}.$$

Pick $\gamma > 0$ and consider the set

$$\Phi := \left\{\phi_\delta : 0 \le \delta < \gamma\right\}.$$

Recall, that in view of Section 3, the proof of Theorem 1.4 has been reduced to showing the truth of (2.5). The above transference principle plays a key role in carrying out this task. With this in mind, the proof of (2.5) splits naturally into three main steps. Let $B \subseteq U$ be an open ball and recall that ϱ is the d-dimensional Lebesgue measure restricted to B.

Step 1: We show that if γ is appropriately chosen, then

$$\varrho\big(\Lambda_H(\phi_\delta)\big) = 0 \quad \forall\, \delta \in [0,\gamma)\,, \tag{5.3}$$

regardless of the choice B. This will be the subject of Sections 6–8.

Step 2: We show that the triplet (H, I, Φ) as defined above satisfies the intersection property. This will be the subject of Section 9.

Step 3: We show that ϱ is contracting with respect to (I, ϕ_δ). This will be the subject of Section 10.

The upshot of successfully carry out these steps, is that on applying Theorem 4.1, we are able to conclude that

$$\varrho\big(\Lambda_I(\phi_\delta)\big) = 0 \quad \forall\, \delta \in [0,\gamma).$$

This in turn implies (2.5), since

$$\limsup \mathcal{L}_{small}(\mathbf{a}, B) \subseteq \Lambda_I(\phi_\delta) \cap B \quad \forall\, \delta > 0.$$

To carry out Step 1 we shall employ dynamical tools. For that, we need to recall a few elementary properties of the so called "good functions" introduced by Kleinbock and Margulis [26].

6 (C,α)-good functions

Let C and α be positive numbers and V be an open subset of $\mathbb{R}^d$. A function $f : V \to \mathbb{R}$ is said to be (C,α)-good on V if for any open ball $B \subseteq V$, and for any $\varepsilon > 0$, one has:

$$\left|\left\{\mathbf{x} \in B : |f(\mathbf{x})| < \varepsilon\right\}\right| \le C\left(\frac{\varepsilon}{\sup_{\mathbf{x}\in B}|f(\mathbf{x})|}\right)^{\alpha} |B|. \tag{6.1}$$

Now consider $\mathbf{f} = (f_1, \ldots, f_n)$, a map from an open subset $U \subseteq \mathbb{R}^d$ to $\mathbb{R}^n$. We will say that $\mathbf{f}$ is good at $\mathbf{x}_0 \in U$ if there exists a neighbourhood $V \subseteq U$ of $\mathbf{x}_0$ and $C, \alpha > 0$ such that any linear combination of $1, f_1, \ldots, f_n$ is (C,α)-*good* on V. The map $\mathbf{f}$ is said to be good if it is good at every point of U. Note that C, α need not be uniform.

We will make use of the following properties of (C,α)-good functions, for example, see [24].

- (G1) If f is (C,α)-good on an open set V, so is λf for all $\lambda \in \mathbb{R}$.
- (G2) If $f_i, i \in I$ are (C,α)-good on V, so is $\sup_{i\in I}|f_i|$.
- (G3) If f is (C,α)-good on V and for some $c_1, c_2 > 0$, $c_1 \le \frac{|f(\mathbf{x})|}{|g(\mathbf{x})|} \le c_2$ for all $\mathbf{x} \in V$, then g is $(C(c_2/c_1)^{\alpha}, \alpha)$-good on V.
- (G4) If f is (C,α)-good on V, it is (C', α')-good on V' for every $C' \ge \max\{C, 1\}$, $\alpha' \le \alpha$ and $V' \subset V$.

One can note that from (G2), it follows that the supremum norm of a vector valued function $\mathbf{f}$ is (C,α)-good whenever each of its components is (C,α)-good. Furthermore, in view of (G3), we can replace the norm by an equivalent one, only affecting C but not α.

The following result provides us with an important class of good functions.

Proposition 6.1. [11, Lemma 3.2] Any polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ of degree not exceeding l is $(C_{d,l}, \frac{1}{dl})$-good on $\mathbb{R}^d$, where $C_{d,l} = \frac{2^{d+1} dl(l+1)^{1/l}}{V_d}$ and V_d is the volume of the unit ball in $\mathbb{R}^d$ with respect to the Euclidean norm. In particular, constant and linear polynomials are $(\frac{2^{d+2}d}{V_d}, \frac{1}{d})$-good on $\mathbb{R}^d$.

The main dynamical tool that we will be exploiting to show (5.3) is commonly known as the "quantitative nondivergence" estimate in the space of unimodular lattices. This constitutes our next section.

7 A Quantitative Nondivergence Estimate

Let W be a finite dimensional real vector space. For a discrete subgroup Γ of W, we set $\Gamma_{\mathbb{R}}$ to be the minimal linear subspace of W containing Γ. A subgroup Γ of Λ is said to be primitive in Λ if $\Gamma = \Gamma_{\mathbb{R}} \cap \Lambda$. We denote the set of all nonzero primitive subgroups of Γ by $\mathcal{L}(\Gamma)$. Let $j := \dim(\Gamma_{\mathbb{R}})$ be the *rank* of Γ. We say that $\mathbf{w} \in \bigwedge^j(W)$ represents Γ if

$$\mathbf{w} = \begin{cases} 1 & \text{if} \quad j = 0 \\ \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j & \text{if} \quad j > 0 \text{ and } \mathbf{v}_1, \ldots, \mathbf{v}_j \text{ is a basis of } \Gamma. \end{cases}$$

In fact, one can easily see that such a representative of Γ is always unique up to a sign.

A function $\nu : \bigwedge(W) \longrightarrow \mathbb{R}_+$ is called *submultiplicative* if

(i) ν is continuous with respect to natural topology on $\bigwedge(W)$,
(ii) $\forall\, t \in \mathbb{R}$ and $\mathbf{w} \in \bigwedge(W)$, $\nu(t\mathbf{w}) = |t|\nu(\mathbf{w})$, and
(iii) $\forall\, \mathbf{u}, \mathbf{w} \in \bigwedge(W)$, $\nu(\mathbf{u} \wedge \mathbf{w}) \leq \nu(\mathbf{u})\nu(\mathbf{w})$.

In view of property (ii) above, without any confusion, we can define $\nu(\Gamma) := \nu(\mathbf{w})$ where $\mathbf{w}$ represents Γ. Armed with the notion of submultiplicative, we are in the position to state the "quantitative nondivergence" estimate that we will require in establishing (5.3).

Theorem 7.1. [11, Theorem 6.2] Let W be a finite dimensional real vector space, Λ a discrete subgroup of W of rank k, and let a ball $B = B(\mathbf{x}_0, r_0) \subset \mathbb{R}^d$ and a continuous map $H : \tilde{B} \to \mathrm{GL}(W)$ be given, where $\tilde{B} := B(\mathbf{x}_0, 3^k r_0)$. Take $C \geq 1$, $\alpha > 0$, $0 < \rho < 1$ and let ν be a submultiplicative function on $\bigwedge(W)$. Assume that for any $\Gamma \in \mathcal{L}(\Lambda)$,

(KM1) the function $\mathbf{x} \mapsto \nu(H(\mathbf{x})\Gamma)$ is (C, α)-good on $\tilde{B}$,
(KM2) $\sup_{\mathbf{x} \in B} \nu(H(\mathbf{x})\Gamma) \geq \rho$,
(KM3) $\forall\, \mathbf{x} \in \tilde{B}$, $\#\{\Gamma \in \mathcal{L}(\Lambda) : \nu(H(\mathbf{x})\Gamma) < \rho\} < \infty$.

Then for every $\varepsilon'' > 0$ we have that

$$\left|\left\{\mathbf{x} \in B \,:\, \nu(H(\mathbf{x})\boldsymbol{\lambda}) < \varepsilon'' \text{ for some } \boldsymbol{\lambda} \in \Lambda\backslash\{0\}\right\}\right| \;<\; k\,(3^d N_d)^k\, C \left(\frac{\varepsilon''}{\rho}\right)^{\alpha} |B|, \tag{7.1}$$

where N_d is the Besicovitch constant for $\mathbb{R}^d$.

8 Proof of (5.3)

Fix a ball $B \subset U$ such that $11B \subset U$. For $t \in \mathbb{Z}_+$ and $\delta \in [0, \gamma)$, we define the set

$$\begin{aligned}\mathcal{A}_t &:= \bigcup_{\alpha \in \mathcal{A}} \big(H_t\big(\alpha, \phi_\delta(t)\big) \cap B\big) \\ &= \left\{ \mathbf{x} \in B : \exists (\mathbf{a}, a_0) \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \times \mathbb{Z} \ \text{s.t.} \ \middle| \ \begin{array}{l} \big|a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\big| < \frac{2 \times 2^{\delta t}}{2^{nt}} \\ \big\|\nabla(\mathbf{x}, \tilde{\mathbf{x}}A) \cdot \mathbf{a}\big\| < 2\sqrt{nd L} \times 2^{\delta t} \times 2^{t/2} \\ \|\mathbf{a}\| < 2^{t+2} \end{array} \right\}\end{aligned}$$

Then, by definition

$$\Lambda_H(\phi_\delta) \cap B \ \subseteq \ \limsup_{t \to \infty} \mathcal{A}_t$$

and so (5.3) follows on showing that

$$\big|\limsup_{t \to \infty} \mathcal{A}_t\big| = 0. \tag{8.1}$$

With this in mind, pick $\beta \in \big(0, \frac{1}{2(n+1)}\big)$ and set

$$\delta' := \frac{2}{2^{nt}}, \qquad K := 2 \times \sqrt{nd L} \times 2^{t/2}, \qquad T := 2^{t+2}, \tag{8.2}$$

$$\varepsilon' := (\delta' K T^{n-1})^{\frac{1}{n+1}} \ = \ \Big(2^{2n}\sqrt{nd L}\Big)^{\frac{1}{n+1}} \frac{1}{2^{t/2(n+1)}}, \tag{8.3}$$

and

$$\varepsilon := 2^{\beta t}\varepsilon' \ = \ \Big(2^{2n}\sqrt{nd L}\Big)^{\frac{1}{n+1}} \frac{2^{\beta t}}{2^{t/2(n+1)}}. \tag{8.4}$$

Furthermore, for $\mathbf{x} \in \mathbb{R}^d$, let

$$u_{\mathbf{x}} := \begin{pmatrix} 1 & 0 & \mathbf{x} & \mathbf{x}A' + \mathbf{a}_0 \\ 0 & I_d & I_d & A' \\ 0 & 0 & I_n & \end{pmatrix} \tag{8.5}$$

and for $t \in \mathbb{Z}_+$, let

$$g_t := \operatorname{diag}\left(\frac{\varepsilon}{\delta'}, \frac{\varepsilon}{K}, \ldots, \frac{\varepsilon}{K}, \frac{\varepsilon}{T}, \ldots, \frac{\varepsilon}{T}\right), \tag{8.6}$$

where $\varepsilon, \delta', T, K$ are defined above and the $\frac{\varepsilon}{K}$ and $\frac{\varepsilon}{T}$ appear d and n times, respectively. Note that these parameters depend on t and some fixed constants. Also, denote by Λ the subgroup of $\mathbb{Z}^{1+d+n}$ consisting of vectors of the form:

$$\Lambda = \left\{ \begin{pmatrix} p \\ 0 \\ \vdots \\ 0 \\ \mathbf{q} \end{pmatrix} : p \in \mathbb{Z}, \mathbf{q} \in \mathbb{Z}^n \right\}. \tag{8.7}$$

Then, it readily follows from the above definitions that

$$\mathcal{A}_t \subseteq \tilde{\mathcal{A}}_t := \left\{ \mathbf{x} \in B : \|g_t u_{\mathbf{x}} \lambda\| < 2^{\delta t}\varepsilon \text{ for some } \lambda \in \Lambda \backslash \{0\} \right\}, \tag{8.8}$$

and so (8.1) follows on showing that

$$\left| \limsup_{t \to \infty} \tilde{\mathcal{A}}_t \right| = 0.$$

In view of the Borel–Cantelli lemma, this will follow on showing that

$$\sum_{t=0}^{\infty} \left| \tilde{\mathcal{A}}_t \right| < \infty. \tag{8.9}$$

With the intention of using Theorem 7.1 to prove (8.9), we take $W = \mathbb{R}^{1+d+n}$ with basis $\mathbf{e}_0, \mathbf{e}_{*1}, \ldots, \mathbf{e}_{*d}, \mathbf{e}_1, \ldots, \mathbf{e}_n$, Λ as given by (8.7) and $H(\mathbf{x}) = g_t u_{\mathbf{x}}$. The submultiplicative function ν on W is chosen as described in [11, §7]. Namely, let W_* be the d-dimensional subspace of W spanned by $\mathbf{e}_{*1}, \ldots, \mathbf{e}_{*d}$ so that Λ given by (8.7) is equal to the intersection of $\mathbb{Z}^{1+d+n}$ and $W_*^{\perp}$. Here we identify $W_*^{\perp}$ with $\mathbb{R}^{n+1}$ canonically. Also, let $\mathcal{W}$ be the ideal of $\bigwedge(W)$ generated by $\bigwedge^2(W_*)$, and let π_* be the orthogonal projection with kernel $\mathcal{W}$. Then $\|\mathbf{w}\|_e$ is defined to be the Euclidean norm of $\pi_*(\mathbf{w})$. In other words, if $\mathbf{w}$ is written as a sum of exterior products of the base vectors $\mathbf{e}_i$ and $\mathbf{e}_{*i}$, to compute $\nu(\mathbf{w})$ we ignore the components containing exterior products of the type $\mathbf{e}_{*i} \wedge \mathbf{e}_{*j}$, $1 \le i \ne j \le d$, and simply take the Euclidean norm of the sum of the remaining components. By definition, it is immediate that $\nu|_W$ agrees with the Euclidean norm.

For appropriately determined quantities C, α, ρ we now validate, one by one, the conditions (KM1)–(KM3) associated with Theorem 7.1. Condition (KM3) can be verified for any $\rho \le 1$ in exactly the same manner as in [11, §7]. For the verification of the remaining conditions, we begin with the explicit computation of the quantity $H(\mathbf{x})\mathbf{w}$

for any $\mathbf{w} \in \bigwedge^k(W_*^\perp)$ and $k = 1, \ldots, n+1$. On writing $\mathbf{x} = (x_1, \ldots, x_d)$ and $(\mathbf{x}, \tilde{\mathbf{x}}A) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$, we see that

(1) $H(\mathbf{x})\,\mathbf{e}_0 = \frac{\varepsilon}{\delta'}\,\mathbf{e}_0$,
(2) $H(\mathbf{x})\,\mathbf{e}_{*i} = \frac{\varepsilon}{K}\,\mathbf{e}_{*i}$ for all $1 \le i \le d$,
(3) $H(\mathbf{x})\,\mathbf{e}_i = \frac{\varepsilon}{\delta'}\,f_i(\mathbf{x})\,\mathbf{e}_0 + \frac{\varepsilon}{K}\,\sum_{j=1}^d \frac{\partial f_j(\mathbf{x})}{\partial x_i}\mathbf{e}_{*j} + \frac{\varepsilon}{T}\,\mathbf{e}_i$ for all $1 \le i \le n$.

Note that each $f_i(\mathbf{x})$ is a polynomial in $x_1, \ldots, x_d$ with degree at most 1 so that each partial derivative $\frac{\partial f_j(\mathbf{x})}{\partial x_i}$ is constant.

8.1 Checking (KM1)

Since $\Lambda = \mathbb{Z}^{1+d+n} \cap W_*^\perp$, any representative $\mathbf{w} \in \bigwedge^k(W)$ of any subgroup of Λ of rank k, $1 \le k \le n+1$, can be written as $\sum_I a_I\mathbf{e}_I$, where each $a_I \in \mathbb{Z}$ and $\mathbf{e}_I = \mathbf{e}_{i_i} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $i_1, \ldots, i_k \in \{0, 1, \ldots, n\}, i_1 < \cdots < i_k$.

Since each component of $\pi_*(H(\mathbf{x})\mathbf{w})$ is a polynomial in $x_1, \ldots, x_d$ with degree at most 1 and in view of Proposition 6.1, each of them is $\left(\frac{2^{d+2}d}{V_d}, \frac{1}{d}\right)$-good on $\tilde{B}$. This implies that the function $\mathbf{x} \mapsto \|\pi_*(H(\mathbf{x})\mathbf{w})\|$ is $\left(\frac{2^{d+2}d}{V_d}, \frac{1}{d}\right)$-good on $\tilde{B}$. As

$$\frac{1}{2^{\frac{1+d+n}{2}}} \;\le\; \frac{\left\|\pi_*(H(\mathbf{x})\mathbf{w})\right\|}{\nu\big(\pi_*(H(\mathbf{x})\mathbf{w})\big)} \;\le\; 1,$$

it follows from properties (G3) and (G4) of good functions that $\nu(\pi_*(H(\mathbf{x})\mathbf{w}))$ is (C, α)-good on $\tilde{B}$ with

$$C := \max\left\{\frac{2^{\left(d+2+\frac{1+d+n}{2d}\right)}d}{V_d}, 1\right\} \text{ and } \alpha := \frac{1}{d}\,. \tag{8.10}$$

This verifies condition (KM1).

8.2 Checking (KM2)

Let Γ be a subgroup of Λ with rank k and $\mathbf{w} \in \bigwedge^k(W_*^\perp)$ represent Γ. We first consider the case $k = n+1$. Thus, $\mathbf{w} = w\,\mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$, where $w \in \mathbb{Z}\backslash\{0\}$. Hence, for any $\mathbf{x} \in B$, it is easily verified that the coefficient of $\mathbf{e}_0 \wedge \mathbf{e}_{*1} \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$ in $\pi_*(H(\mathbf{x})\mathbf{w})$ is

$$w\frac{\varepsilon^{n+1}}{\delta' K T^{n-1}}.$$

It now follows via (8.2)–(8.4), that

$$\sup_{\mathbf{x}\in B} \nu\big(H(\mathbf{x})\Gamma\big) = \sup_{\mathbf{x}\in B} \nu\big(H(\mathbf{x})\mathbf{w}\big) \geq \sup_{\mathbf{x}\in B} \big\|\pi_*(H(\mathbf{x})\mathbf{w})\big\|$$
$$\geq \left| w \frac{\varepsilon^{n+1}}{\delta' K T^{n-1}} \right|$$
$$= |w| 2^{\beta(n+1)t} \frac{(\varepsilon')^{n+1}}{\delta' K T^{n-1}}$$
$$= |w| 2^{\beta(n+1)t} \geq 1. \tag{8.11}$$

Thus, when $k = n+1$ condition (KM2) is valid for any $0 < \rho < 1$.

Assume now that $1 \leq k \leq n$. To bound the norm of $\|\pi_*(H(\mathbf{x})\mathbf{w})\|$ from below, we will proceed along the lines of [19, §5.3] using a technique from [25]. As observed in [19, §5.3], for any $\mathbf{x} \in B$

$$\big\|\pi_*(H(\mathbf{x})\mathbf{w})\big\| \geq \big\|\tilde{g}_t \tilde{u}_{\mathbf{x}} \mathbf{w}\big\|,$$

where

$$\tilde{u}_{\mathbf{x}} = \begin{pmatrix} 1 & \mathbf{x} & \tilde{\mathbf{x}} A \\ 0 & I_n & \end{pmatrix} \quad \text{and} \quad \tilde{g}_t = \operatorname{diag}\Big(\frac{\varepsilon}{\delta'}, \frac{\varepsilon}{T}, \ldots, \frac{\varepsilon}{T}\Big). \tag{8.12}$$

Hence,

$$\sup_{\mathbf{x}\in B} \nu\big(H(\mathbf{x})\Gamma\big) = \sup_{\mathbf{x}\in B} \nu\big(H(\mathbf{x})\mathbf{w}\big) \geq \sup_{\mathbf{x}\in B} \big\|\pi_*(H(\mathbf{x})\mathbf{w})\big\|$$
$$\geq \sup_{\mathbf{x}\in B} \big\|\tilde{g}_t \tilde{u}_{\mathbf{x}} \mathbf{w}\big\|. \tag{8.13}$$

Thus, the name of the game is to bound $\sup_{\mathbf{x}\in B} \|\tilde{g}_t \tilde{u}_{\mathbf{x}} \mathbf{w}\|$ from below. It follows from (4.6) in [25], that

$$\sup_{\mathbf{x}\in B} \big\|\tilde{g}_t \tilde{u}_{\mathbf{x}} \mathbf{w}\big\| \;\geq\; \frac{1}{2^{\frac{n+1}{2}}} \max\left\{ \left(\frac{\varepsilon^k}{\delta' T^{k-1}}\right) \sup_{\mathbf{x}\in B} \big\|(\mathbf{x}, \tilde{\mathbf{x}} A)\, \mathbf{c}(\mathbf{w})\big\|,\; \left(\frac{\varepsilon}{T}\right)^k \big\|\pi(\mathbf{w})\big\| \right\}, \tag{8.14}$$

where $\mathbf{c}$ is the function given by (1.13), π is the projection from $\bigwedge(W_*^{\perp})$ to $\bigwedge(W_{1\to n})$, and $W_{1\to n}$ stands for the span of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Now, recall that

$$(\mathbf{x}, \tilde{\mathbf{x}} A) = \tilde{\mathbf{x}} R_A,$$

where R_A is given by (1.11). Therefore, we can replace $\sup_{\mathbf{x}\in B} \|(\mathbf{x}, \tilde{\mathbf{x}} A)\mathbf{c}(\mathbf{w})\|$ in the above norm calculation by $\sup_{\mathbf{x}\in B} \|\tilde{\mathbf{x}} R_A \mathbf{c}(\mathbf{w})\|$. As the functions $1, x_1, \ldots, x_d$ are linearly independent over $\mathbb{R}$ on B, the map

$$\mathbf{v} \mapsto \sup_{\mathbf{x}\in B} \|\tilde{\mathbf{x}}\mathbf{v}\|$$

defines a norm on $(\bigwedge(W_{1\to n}))^{d+1}$, which must be equivalent to the supremum norm on $(\bigwedge(W_{1\to n}))^{d+1}$. Thus, there is a constant $K_2 > 0$ depending on d, n and B, such that

$$\sup_{\mathbf{x}\in B} \left\|\tilde{\mathbf{x}}\, R_A\ \mathbf{c}(\mathbf{w})\right\| \ \geq\ K_2 \left\|R_A\ \mathbf{c}(\mathbf{w})\right\|,$$

and consequently, via (8.14), that

$$\sup_{\mathbf{x}\in B} \left\|\tilde{g}_t\tilde{u}_{\mathbf{x}}\mathbf{w}\right\| \ \geq\ \frac{1}{2^{\frac{n+1}{2}}} \max\left\{\left(\frac{\varepsilon^k}{\delta' T^{k-1}}\right) K_2 \left\|R_A\ \mathbf{c}(\mathbf{w})\right\|, \left(\frac{\varepsilon}{T}\right)^k \left\|\pi(\mathbf{w})\right\|\right\}. \tag{8.15}$$

To continue, we consider two separate cases depending on the size of the rank k. We first note that from Lemma 5.1 in [25], we get that for any $n-d < k \leq n$ for all but finitely many $\mathbf{w} \in \bigwedge^k(\Lambda)$ we have that $\|R_A\ \mathbf{c}(\mathbf{w})\| \geq 1$. Also, note that $\|R_A\ \mathbf{c}(\mathbf{w})\|$ does not vanish, as otherwise if $\|R_A\ \mathbf{c}(\mathbf{w}_0)\|$ were zero, then, by the linearity of the map $\mathbf{c}(\mathbf{w})$, we would get $\|R_A\ \mathbf{c}(\lambda\mathbf{w})\| = 0$ for all integers λ, contrary to what we have already seen. Consequently, there is a constant $K_3 > 0$ depending only on A, such that

$$\left\|R_A\ \mathbf{c}(\mathbf{w})\right\| \geq K_3. \tag{8.16}$$

Therefore, by (8.15), we get that

$$\sup_{\mathbf{x}\in B} \left\|\tilde{g}_t\tilde{u}_{\mathbf{x}}\mathbf{w}\right\| \geq \frac{K_2K_3}{2^{\frac{n+1}{2}}}\left(\frac{\varepsilon^k}{\delta' T^{k-1}}\right). \tag{8.17}$$

It follows from (8.2)–(8.4), that

$$\begin{aligned}
\frac{\varepsilon^k}{\delta' T^{k-1}} &= \left(2^{2n}\sqrt{n\mathrm{d}L}\right)^{\frac{k}{n+1}} \ \frac{1}{2^{\left(\frac{1}{2(n+1)}-\beta\right)kt}}\ \frac{2^{nt}}{2}\ \frac{1}{2^{(t+2)(k-1)}} \\
&\geq \min_{n-d<k\leq n} \left(2^{2n}\sqrt{n\mathrm{d}L}\right)^{\frac{k}{n+1}} \ \frac{1}{2^{\left(\frac{1}{2(n+1)}-\beta\right)nt}}\ \frac{2^{nt}}{2}\ \frac{1}{2^{(t+2)(n-1)}} \\
&= \frac{1}{2^{2n-1}} \ \min_{n-d<k\leq n} \left(2^{2n}\sqrt{n\mathrm{d}L}\right)^{\frac{k}{n+1}}\ 2^{\left(1-\left(\frac{1}{2(n+1)}-\beta\right)n\right)t}.
\end{aligned}$$

On picking β such that

$$\frac{1}{2(n+1)} - \frac{1}{n} \ < \ \beta \ < \ \frac{1}{2(n+1)}, \tag{8.18}$$

we obtain via (8.17), that for all subgroups Γ of Λ with rank $n-d < k \le n$

$$\sup_{\mathbf{x}\in B} \left\|\tilde{g}_t \tilde{u}_{\mathbf{x}} \mathbf{w}\right\| \ \ge \ \frac{K_2 K_3}{2^{\frac{5n-1}{2}}} \min_{n-d<k\le n} \left(2^{2n}\sqrt{n d L}\right)^{\frac{k}{n+1}}. \tag{8.19}$$

We now obtain an analogous lower bound result for subgroups Γ of Λ with rank $1 \le k \le n-d$. In this case, a consequence of (1.15) is that there exist constants $0 < \theta', K_4 < 1$, depending only on A, such for any $\mathbf{w} \in \bigwedge^k(\Lambda)$

$$\left\|R_A\ \mathbf{c}(\mathbf{w})\right\| \ \ge \ K_4 \left\|\pi_\bullet(\mathbf{w})\right\|^{-\frac{(n-\theta')+1-k}{k}}. \tag{8.20}$$

Also, $\|\pi(\mathbf{w})\| \ge \|\pi_\bullet(\mathbf{w})\|$ if $1 \le k \le n-d$. Therefore, it follows from (8.15) that

$$\begin{aligned}
\sup_{\mathbf{x}\in B} \left\|\tilde{g}_t \tilde{u}_{\mathbf{x}} \mathbf{w}\right\| \quad &\ge \quad \max\left\{\left(\frac{\varepsilon^k}{\delta' T^{k-1}}\right) K_2 K_4 \left\|\pi_\bullet(\mathbf{w})\right\|^{-\frac{(n-\theta')+1-k}{k}},\ \left(\frac{\varepsilon}{T}\right)^k \left\|\pi_\bullet(\mathbf{w})\right\|\right\} \\
&\ge \quad \kappa \left(\frac{\varepsilon}{T}\right)^k,
\end{aligned} \tag{8.21}$$

where κ is the solution to the equation

$$\frac{K_2 K_4 T}{\delta'} \, y^{-\frac{(n-\theta')+1-k}{k}} \ = \ y. \tag{8.22}$$

In other words, $\kappa := (K_2K_4)^{\frac{k}{n-\theta'+1}} \left(\frac{T}{\delta'}\right)^{\frac{k}{n-\theta'+1}}$ and so it follows from (8.2)–(8.4) that

$$\begin{aligned}
\kappa \left(\frac{\varepsilon}{T}\right)^k &= (K_2K_4)^{\frac{k}{n-\theta'+1}} \left(\frac{T}{\delta'}\right)^{\frac{k}{n-\theta'+1}} \left(\frac{\varepsilon}{T}\right)^k \\
&= (K_2K_4)^{\frac{k}{n-\theta'+1}} 2^{\frac{k}{n-\theta'+1}} 2^{\frac{(n+1)kt}{n-\theta'+1}} \left(2^{2n}\sqrt{n d L}\right)^{\frac{k}{n+1}} \frac{1}{2^{\left(\frac{1}{2(n+1)}-\beta\right)kt}} \frac{1}{2^{(t+2)k}} \\
&= (K_2K_4)^{\frac{k}{n-\theta'+1}} 2^{\frac{k}{n-\theta'+1}} \frac{1}{2^{2k}} \left(2^{2n}\sqrt{n d L}\right)^{\frac{k}{n+1}} 2^{\left(\left(\frac{n+1}{n-\theta'+1}-1\right)-\left(\frac{1}{2(n+1)}-\beta\right)\right)kt} \\
&= (K_2K_4)^{\frac{k}{n-\theta'+1}} \left(2^{2n}\sqrt{n d L}\right)^{\frac{k}{n+1}} 2^{\left(\frac{1}{n-\theta'+1}-2\right)k} 2^{\left(\left(\frac{n+1}{n-\theta'+1}-1\right)-\left(\frac{1}{2(n+1)}-\beta\right)\right)kt}.
\end{aligned}$$

On redefining β if necessary, namely so that both (8.18) and

$$\frac{1}{2(n+1)} - \frac{\theta'}{(n-\theta')+1} \;<\; \beta \;<\; \frac{1}{2(n+1)}, \tag{8.23}$$

hold, it follows that

$$\kappa\left(\frac{\varepsilon}{T}\right)^k \geq K_5 := \min_{1\leq k\leq n-d} (K_2K_4)^{\frac{k}{n-\theta'+1}} \left(2^{2n}\sqrt{n d L}\right)^{\frac{k}{n+1}} 2^{\left(\frac{1}{n-\theta'+1}-2\right)k}.$$

Note that (8.23) has a solution β since $0 < \theta' < 1$. This together with (8.21) implies that for all subgroups Γ of Λ with rank $1 \leq k \leq n-d$, we have that

$$\sup_{\mathbf{x}\in B} \left\|\tilde{g}_t \tilde{u}_{\mathbf{x}} \mathbf{w}\right\| \;\geq\; \frac{1}{2^{\frac{n+1}{2}}} K_5. \tag{8.24}$$

On combining (8.11), (8.13), (8.19), and (8.24), we have verified condition (KM2) with

$$\rho \;:=\; \min\left\{\frac{1}{2},\; \frac{K_2K_3}{2^{\frac{5n-1}{2}}} \min_{n-d<k\leq n} \left(2^{2n}\sqrt{n d L}\right)^{\frac{k}{n+1}},\; \frac{1}{2^{\frac{n+1}{2}}} K_5\right\}. \tag{8.25}$$

We are now in the position to apply Theorem 7.1 to establish the desired convergent sum statement (8.9).

8.3 The proof of (8.9)

With the choice of $\beta \in (0, 1/2)$ made in the Section 8.2, let

$$0 < \gamma < \frac{1}{2(n+1)} - \beta. \tag{8.26}$$

Clearly, $\gamma > 0$ and note that for any $\delta \in [0, \gamma)$

$$\tilde{\mathcal{A}}_t \subseteq \left\{\mathbf{x} \in B \,:\, \nu\big(H(\mathbf{x})\boldsymbol{\lambda}\big) < \sqrt{1+d+n}\, 2^{\delta t}\, \varepsilon \text{ for some } \boldsymbol{\lambda} \in \Lambda\backslash\{0\}\right\}.$$

Here we make use of the fact that $\nu|_W$ coincides with the Euclidean norm on W. Now on applying Theorem 7.1 with $\varepsilon'' := \sqrt{1+d+n}\, 2^{\delta t}\varepsilon$, where ε is given by (8.4), and C, α, and ρ

are as given in (8.10) and (8.25), we have

$$
\begin{aligned}
|\tilde{\mathcal{A}}_t| &\leq \left|\left\{\mathbf{x} \in B \,:\, \nu(H(\mathbf{x})\boldsymbol{\lambda}) < \sqrt{1+d+n}\, 2^{\delta t}\, \varepsilon \ \text{ for some } \boldsymbol{\lambda} \in \Lambda\backslash\{0\}\right\}\right| \\
&\leq (n+1)(3^d N_d)^{n+1} C(1+d+n)^{\frac{1}{2d}} \left(\frac{\varepsilon}{\rho}\right)^{\frac{1}{d}} |B| \\
&\leq (n+1)(3^d N_d)^{n+1} C(1+d+n)^{\frac{1}{2d}} \frac{1}{\rho^{\frac{1}{d}}} \left(2^{2n}\sqrt{n\mathrm{d}L}\right)^{\frac{1}{d(n+1)}} \frac{1}{2^{\left(\frac{\frac{1}{2(n+1)}-(\beta+\delta)}{d}\right)t}} |B|. \qquad (8.27)
\end{aligned}
$$

As $\delta < \gamma$, it follows via (8.26) that $\delta + \beta < \frac{1}{2(n+1)}$, and so

$$
\sum_{t=0}^{\infty} |\tilde{\mathcal{A}}_t| \ll \sum_{t=0}^{\infty} 2^{-\left(\frac{\frac{1}{2(n+1)}-(\beta+\delta)}{d}\right)t} < \infty.
$$

By the Borel–Cantelli lemma, this establishes (8.9), as desired.

9 Verification of the Intersection Property for (H, I, Φ)

Let γ be given by (8.26) and let $\delta \in [0, \gamma)$. Suppose $t \in \mathcal{T} := \mathbb{Z}_+$ is such that $t(n-\delta) \geq 1$ and $\alpha := (\mathbf{a}, a_0), \alpha' := (\mathbf{a}', a_0') \in \mathcal{A}$ with $\alpha \neq \alpha'$. Recall, $\mathcal{A} := (\mathbb{Z}^n \setminus \{\mathbf{0}\}) \times \mathbb{Z}$. Then, for any $\mathbf{x} \in I_t(\alpha, \phi_\delta(t)) \cap I_t(\alpha', \phi_\delta(t))$, it is easily verified that

$$
\begin{cases}
\left|(a_0 - a_0') + (\mathbf{x}, \tilde{\mathbf{x}}A)(\mathbf{a} - \mathbf{a}')\right| < \frac{2\times 2^{\delta t}}{2^{nt}} \\
\left\|\nabla(\mathbf{x}, \tilde{\mathbf{x}}A) \cdot (\mathbf{a} - \mathbf{a}')\right\| < 2\sqrt{ndL} \times 2^{\delta t} \times 2^{t/2} \\
\left\|(\mathbf{a} - \mathbf{a}')\right\| < 2^{t+2}
\end{cases} . \qquad (9.1)
$$

Suppose for the moment that $\mathbf{a} = \mathbf{a}'$. Then $a_0 \neq a_0'$ as $\alpha \neq \alpha'$. This implies, in view of the first inequality of (9.1), that $1 \leq |(a_0 - a_0')| < \frac{1}{2^{t(n-\delta)-1}} \leq 1$, which is a contradiction. Thus, $\mathbf{a} \neq \mathbf{a}'$ and so $(\mathbf{a} - \mathbf{a}', a_0 - a_0') \in \mathcal{A}$. The upshot of this together with (9.1) is that $\mathbf{x} \in H_t(\alpha'', \phi_\delta(t))$ with $\alpha'' = (\mathbf{a} - \mathbf{a}', a_0 - a_0')$. This establishes (4.1) with $\phi^* = \phi = \phi_\delta$ and thereby verifies the desired intersection property associated with the Inhomogeneous Transference Principle.

10 Verification of the Contraction Property of ϱ

With reference to Section 5, recall that showing ϱ is contracting with respect to (I, ϕ_δ) is the third and final step in establishing Theorem 1.4. We start by observing that in view of [24, Corollary 3.3] and the fact that the inhomogeneous function θ restricted to $\mathscr{H}$ is analytic (it is worth pointing out that this is the only point in the proof of Theorem 1.4 where we use the fact that $\theta|_{\mathscr{H}}$ is analytic), the functions

$$\mathbf{x} \mapsto \left|\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\right|$$

and

$$\mathbf{x} \mapsto \left\|\nabla(\hat{\theta}(\mathbf{x}) + (\mathbf{x}, \tilde{\mathbf{x}}A)\cdot\mathbf{a})\right\|$$

defined on U are good at every point of U. Now pick a point $\mathbf{x}_0 \in U$. On using property (G4) of good functions if necessary, we can choose an open ball B with centre at $\mathbf{x}_0$ and two positive constants $\mathfrak{C}, \alpha_0$ such that the above two functions are $(\mathfrak{C}, \alpha_0)$-*good* on $11B \subseteq U$. Throughout this section we fix such a ball B.

For each $t \in \mathfrak{T}$ and $\alpha \in \mathcal{A}$, consider the function $\mathbf{F}_{t,\alpha} : U \to \mathbb{R}$ given by

$$\mathbf{F}_{t,\alpha}(\mathbf{x}) := \max\left\{ \begin{array}{r} 2^{nt}\sqrt{n\mathrm{d}L}\; 2^{t/2}\left|\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\right|, \\ \left\|\nabla(\hat{\theta}(\mathbf{x}) + (\mathbf{x}, \tilde{\mathbf{x}}A)\cdot\mathbf{a})\right\| \end{array} \right\},$$

where L is given by (2.3). It follows at once, from the properties of good functions, that for each $t \in \mathfrak{T}$ and $\alpha \in \mathcal{A}$ we have that

$$\mathbf{F}_{t,\alpha} \text{ is } (\mathfrak{C}, \alpha_0)\text{-}good \text{ on } 11B. \tag{10.1}$$

Next, observe that for any $\eta \in \mathbb{R}_+$ the first two inequalities appearing in (5.1) are equivalent to the following single inequality

$$\mathbf{F}_{t,\alpha}(\mathbf{x}) < \eta\sqrt{n\mathrm{d}L}\; 2^{t/2}.$$

Hence, for any $t \in \mathfrak{T}, \alpha = (\mathbf{a}, a_0) \in \mathcal{A}$ and $\eta \in \mathbb{R}_+$

$$I_t(\alpha, \eta) = \left\{\mathbf{x} \in U : \mathbf{F}_{t,\alpha}(\mathbf{x}) < \eta\sqrt{n\mathrm{d}L}\; 2^{t/2}\right\} \quad \text{if} \quad 2^t \le \|\mathbf{a}\| < 2^{t+1}, \tag{10.2}$$

and $I_t(\alpha, \eta) = \emptyset$ otherwise. For any $\delta \in [0, \gamma)$, consider the function $\phi_\delta^+ : \mathfrak{T} \to \mathbb{R}_+$ given by

$$\phi_\delta^+(t) := 2^{\frac{\delta+\gamma}{2}t}.$$

Clearly, $\phi_\delta^+ \in \Phi$. Also, for any $t \in \mathfrak{T}$ we have that

$$I_t\big(\alpha, \phi_\delta(t)\big) \subseteq I_t\big(\alpha, \phi_\delta^+(t)\big). \tag{10.3}$$

In order to establish the desired contracting property, for all but finitely many $t \in \mathfrak{T}$ and all $\alpha \in \mathcal{A}$, we need to ensure the existence of a collection $\mathcal{C}_{t,\alpha}$ of balls $\mathfrak{B}$ centred in $\overline{B}$ and an appropriate sequence $\{k_t\}_{t\in\mathfrak{T}}$ of positive numbers satisfying (4.2), (4.3), and (4.4) with $\phi = \phi_\delta$ and $\phi^+ = \phi_\delta^+$. With this in mind, let $(t, \alpha) \in \mathfrak{T} \times \mathcal{A}$ and suppose that $I_t(\alpha, \phi_\delta(t)) = \emptyset$. Then the collection $\mathcal{C}_{t,\alpha} = \emptyset$ obviously suffices. Thus, we can assume that $I_t(\alpha, \phi_\delta(t)) \neq \emptyset$ and in view of (10.2), it follows that

$$I_t\big(\alpha, \phi_\delta^+(t)\big) \cap B \subseteq \left\{\mathbf{x} \in B : \left|\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\right| < \frac{1}{2^{\left(n - \frac{\delta+\gamma}{2}\right)t}}\right\}. \tag{10.4}$$

Assume for the moment that $\hat{\theta}$ is a linear map given by (1.16). Then, by (1.22) and (1.17), we have that $\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}$ is a linear combination of $x_1, \ldots, x_d$ with at least one of the coefficient being $\gg 2^{t(-n+\gamma')}$ in absolute value, where $0 < \gamma' < n - \omega(A; \boldsymbol{\theta})$. Hence,

$$\sup_{\mathbf{x}\in B} \left|\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\right| \gg 2^{t(-n+\gamma')},$$

where the implied constant will not depend on t. Therefore, in view of (10.4), choosing γ within (8.26) so that we additionally meet the inequalities

$$0 < \gamma < \gamma' \tag{10.5}$$

ensures that

$$I_t\big(\alpha, \phi_\delta^+(t)\big) \cap B \subsetneq B \quad \forall\, t \geq t_0, \tag{10.6}$$

where $t_0 \in \mathbb{N}$ is a sufficiently large constant.

Now consider the case $\hat{\theta}$ is not a linear function. Then,

$$\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a} = \hat{\theta}(\mathbf{x}) + \tilde{\mathbf{x}}(A\mathbf{a}' + \mathbf{a}''), \tag{10.7}$$

where $\mathbf{a}' = (a_{d+1}, \ldots, a_n)^t$ and $\mathbf{a}'' = (a_0, \ldots, a_d)^t$. Thus, (10.7) is a linear combination of the functions 1, x_1, ..., x_d, $\hat{\theta}(\mathbf{x})$, which are linearly independent over $\mathbb{R}$. Therefore, (10.7) is not identically zero. Furthermore, the vector of the coefficients of this linear combination is obviously of norm at least 1. Hence,

$$\inf_{(\mathbf{a},a_0)\in\mathbb{R}^{n+1}} \sup_{\mathbf{x}\in B} \left|\hat{\theta}(\mathbf{x}) + a_0 + (\mathbf{x}, \tilde{\mathbf{x}}A)\mathbf{a}\right| \geq \inf_{\|\boldsymbol{\eta}\|=1} \sup_{\mathbf{x}\in B} \left|\eta_0 + \eta_1 x_1 + \cdots + \eta_d x_d + \eta_{d+1}\hat{\theta}(\mathbf{x})\right| > 0, \tag{10.8}$$

where $\boldsymbol{\eta} = (\eta_0, \ldots, \eta_{d+1})$ and the latter quantity is strictly positive since we take the infimum of a positive continuous function over a compact set (the unit sphere). By (10.4) and (10.8), we once again ensure that (10.6) holds for a sufficiently large choice of t_0.

By (10.3) and the fact that $I_t(\alpha, \phi_\delta^+(t))$ is open, for any $\mathbf{x} \in I_t(\alpha, \phi_\delta(t)) \cap \overline{B}$, there is a ball $\mathfrak{B}'(\mathbf{x})$ centred at $\mathbf{x}$ such that

$$\mathfrak{B}'(\mathbf{x}) \subseteq I_t\left(\alpha, \phi_\delta^+(t)\right) . \tag{10.9}$$

On combining (10.6), (10.9) and the fact that B is bounded, we find that there exists a scaling factor $\tau \geq 1$ such that the ball $\mathfrak{B}(\mathbf{x}) := \tau\mathfrak{B}'(\mathbf{x})$ satisfies

$$\mathfrak{B}(\mathbf{x}) \cap \overline{B} \ \subseteq\ I_t\left(\alpha, \phi_\delta^+(t)\right) \cap \overline{B} \ \subsetneq\ 5\mathfrak{B}(\mathbf{x}) \cap \overline{B} \tag{10.10}$$

and

$$5\mathfrak{B}(\mathbf{x}) \ \subset\ 11\,B. \tag{10.11}$$

For $t \geq t_0$ and $\alpha \in \mathcal{A}$, we now let

$$\mathcal{C}_{t,\alpha} := \left\{\mathfrak{B}(\mathbf{x}) : \mathbf{x} \in I_t\left(\alpha, \phi_\delta(t)\right) \cap \overline{B}\right\}.$$

Then by construction and the left hand side (l.h.s) of (10.10), any such collection of balls automatically satisfies conditions (4.2) and (4.3) with $\phi = \phi_\delta$ and $\phi^+ = \phi_\delta^+$. Regarding condition (4.4), we proceed as follows.

Let $\mathfrak{B} \in \mathcal{C}_{t,\alpha}$. By (10.2) and the right hand side of (10.10), we have that

$$\sup_{\mathbf{x}\in 5\mathfrak{B}} \mathbf{F}_{t,\alpha}(\mathbf{x}) \geq \sup_{\mathbf{x}\in 5\mathfrak{B}\cap\overline{B}} \mathbf{F}_{t,\alpha}(\mathbf{x}) \geq \sqrt{ndL}\, 2^{\frac{\delta+\gamma}{2}t}\, 2^{t/2}\,. \tag{10.12}$$

On the other hand,

$$\sup_{\mathbf{x}\in 5\mathfrak{B}\cap I_t(\alpha,\phi_\delta(t))} \mathbf{F}_{t,\alpha}(\mathbf{x}) \leq \sqrt{ndL}\, 2^{\delta t}\, 2^{t/2}\,. \tag{10.13}$$

On combining (10.12) and (10.13), it follows that

$$\sup_{\mathbf{x}\in 5\mathfrak{B}\cap I_t(\alpha,\phi_\delta(t))} \mathbf{F}_{t,\alpha}(\mathbf{x}) \ \leq\ 2^{\delta t}\, \frac{1}{2^{\frac{\delta+\gamma}{2}t}} \sup_{\mathbf{x}\in 5\mathfrak{B}} \mathbf{F}_{t,\alpha}(\mathbf{x}) \ =\ \frac{1}{2^{\frac{\gamma-\delta}{2}t}} \sup_{\mathbf{x}\in 5\mathfrak{B}} \mathbf{F}_{t,\alpha}(\mathbf{x}).$$

This together with (10.11) and (10.1), implies that for any $t \geq t_0$ and $\alpha \in \mathcal{A}$

$$\begin{aligned} \varrho\big(5\mathfrak{B} \cap I_t\big(\alpha, \phi_\delta(t)\big)\big) &\leq \big|5\mathfrak{B} \cap I_t\big(\alpha, \phi_\delta(t)\big)\big| \\ &\leq \left|\left\{\mathbf{x} \in 5\mathfrak{B} : \mathbf{F}_{t,\alpha}(\mathbf{x}) \leq \frac{1}{2^{\frac{\gamma-\delta}{2}t}} \sup_{\mathbf{x}\in 5\mathfrak{B}} \mathbf{F}_{t,\alpha}(\mathbf{x})\right\}\right| \\ &\leq \tilde{C}\, \frac{1}{2^{\frac{\gamma-\delta}{2}\alpha_0 t}}\ |5\mathfrak{B}|, \end{aligned} \tag{10.14}$$

where $\tilde{C} > 0$ is some constant depending on B. On using (10.11) and the fact that $\mathfrak{B}$ is centred in $\overline{B}$, we have that $|5\mathfrak{B}| \leq c_d \varrho(5\mathfrak{B})$ for some constant c_d depending on d only. Hence, (10.14) implies that for all but finitely many $t \in \mathcal{T}$

$$\varrho\big(5\mathfrak{B} \cap I_t\big(\alpha, \phi_\delta(t)\big)\big) \leq c_d \tilde{C}\, 2^{-\frac{\gamma-\delta}{2}\alpha_0 t}\ \varrho(5\mathfrak{B}).$$

This verifies condition (4.4) with $\phi = \phi_\delta$ and

$$k_t := c_d \tilde{C}\, 2^{-\frac{\gamma-\delta}{2}\alpha_0 t}.$$

Furthermore, it is easily seen that $\sum_{t\in\mathcal{T}} k_t < \infty$ and thus all the conditions of the contracting property are satisfied for the collection $\mathcal{C}_{t,\alpha}$ as defined above.

11 The Divergence Theory: Proof of Theorem 1.5

The proof of Theorem 1.5 makes use of the following statement, which is a special and simplified version of Theorem 3 appearing in [2].

Theorem 11.1. Let $\mathcal{M} := \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in U\} \subseteq \mathbb{R}^n$ be a manifold of dimension d parameterised by a smooth map $\mathbf{f} : U \to \mathbb{R}^n$ defined on a ball $U \subseteq \mathbb{R}^d$. Suppose there exists an absolute constant $C_0 \geq 1$ such that for any ball B with $2B \subseteq U$ and any $\kappa \in (0, 1)$, we have that

$$\left|\left\{\mathbf{x} \in B : \exists\,(\mathbf{a}, a_0) \in \mathbb{Z}^n \setminus \{\mathbf{0}\} \times \mathbb{Z} \text{ s.t. } \left| \begin{array}{l} |a_0 + \mathbf{f}(\mathbf{x}) \cdot \mathbf{a}| < \frac{\kappa}{Q^n} \\ \|\mathbf{a}\| \leq Q \end{array} \right.\right\}\right| \leq C_0\, \kappa\, |B| \tag{11.1}$$

for all sufficiently large Q. Let ψ be an approximation function and $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\theta|_{\mathcal{M}} \in C^2$. Let $s > d-1$ and suppose that

$$\sum_{\mathbf{a}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \|\mathbf{a}\| \left(\frac{\psi(\|\mathbf{a}\|^n)}{\|\mathbf{a}\|}\right)^{s+1-d} = \infty\,. \tag{11.2}$$

Then

$$\mathcal{H}^s\big(\mathcal{W}_n^{\theta}(\psi)\cap\mathcal{M}\big) = \mathcal{H}^s(\mathcal{M})\,. \tag{11.3}$$

Note that, by the monotonicity of ψ, (11.2) is equivalent to (1.23). Hence, armed with Theorem 11.1, the proof of Theorem 1.5 reduces to establishing (11.1) with $U \subset \mathbb{R}^d$ being an open subset, $\mathcal{M} = \mathscr{H}$ and $\mathbf{f}$ given by (1.9). With this in mind, for any ball B such that $11B \subseteq U$, any $\kappa \in (0,1)$ and $Q > 1$, let

$$\mathcal{L}^1(B,\kappa,Q) := \left\{ \mathbf{x}\in B : \exists(\mathbf{a},a_0)\in\mathbb{Z}^n\setminus\{\mathbf{0}\}\times\mathbb{Z}\ \text{s.t.} \left| \begin{array}{r} \big|a_0 + (\mathbf{x},\tilde{\mathbf{x}}A)\cdot\mathbf{a}\big| < \frac{\kappa}{Q^n} \\ \big\|\nabla(\mathbf{x},\tilde{\mathbf{x}}A)\cdot\mathbf{a}\big\| \geq \frac{\sqrt{nd\|\mathbf{a}\|}}{2r} \\ \|\mathbf{a}\| \leq Q \end{array} \right. \right\} \tag{11.4}$$

and

$$\mathcal{L}^2(B,\kappa,Q) := \left\{ \mathbf{x}\in B : \exists(\mathbf{a},a_0)\in\mathbb{Z}^n\setminus\{\mathbf{0}\}\times\mathbb{Z}\ \text{s.t.} \left| \begin{array}{r} \big|a_0 + (\mathbf{x},\tilde{\mathbf{x}}A)\cdot\mathbf{a}\big| < \frac{\kappa}{Q^n} \\ \big\|\nabla(\mathbf{x},\tilde{\mathbf{x}}A)\cdot\mathbf{a}\big\| < \frac{\sqrt{nd\|\mathbf{a}\|}}{2r} \\ \|\mathbf{a}\| \leq Q \end{array} \right. \right\}. \tag{11.5}$$

We note that the set appearing in (11.1) is contained in the union of the "large derivative" set $\mathcal{L}^1(B,\kappa,Q)$ and the "small derivative" set $\mathcal{L}^2(B,\kappa,Q)$. Thus,

$$\text{l.h.s of (11.1)} \ \leq\ \big|\mathcal{L}^1(B,\kappa,Q)\big| \ +\ \big|\mathcal{L}^2(B,\kappa,Q)\big|. \tag{11.6}$$

As in the proof of Theorem 1.4, estimating the measure of the large derivative set is relatively easy and makes use of Proposition 3.1. To begin with, observe that

$$\mathcal{L}^1(B,\kappa,Q) \ =\ \bigcup_{\mathbf{a}\in\mathbb{Z}^n,\,0<\|\mathbf{a}\|\leq Q} \mathcal{L}^1(\mathbf{a},B,\kappa,Q),$$

where for any $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$,

$$\mathcal{L}^1(\mathbf{a}, B, \kappa, Q) := \left\{ \mathbf{x} \in B : \left| \begin{array}{c} \left|a_0 + (\mathbf{x}, \tilde{\mathbf{x}} A) \cdot \mathbf{a}\right| < \frac{\kappa}{Q^n} \\ \left\| \nabla(\mathbf{x}, \tilde{\mathbf{x}} A) \cdot \mathbf{a} \right\| \geq \frac{\sqrt{nd\|\mathbf{a}\|}}{2r} \end{array} \right. \right\}.$$

Now fix $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and with reference to Proposition 3.1, let $F(\mathbf{x}) = (\mathbf{x}, \tilde{\mathbf{x}}A) \cdot \mathbf{a}$ for $\mathbf{x} \in 2B$ and $\delta' = \kappa\, Q^{-n}$. By definition,

$$M \;\geq\; \frac{1}{4r^2},$$

where M is given in (3.2) and so it follows from Proposition 3.1 that

$$\left|\mathcal{L}^1(\mathbf{a}, B, \kappa, Q)\right| \;\leq\; K_d\, \frac{\kappa}{Q^n}\, |B|.$$

In turn, this implies that

$$\left|\mathcal{L}^1(B, \kappa, Q)\right| \;\leq\; K_d\, \frac{\kappa}{Q^n}(2Q+1)^n\, |B| \;\leq\; 3^n K_d\, \kappa\, |B|. \tag{11.7}$$

We now turn our attention to estimating the measure of the small derivative set $\mathcal{L}^2(B, \kappa, Q)$. Let t be the unique integer satisfying $2^t \leq Q < 2^{t+1}$ and δ satisfy $0 < \delta < \gamma$, where γ is as defined earlier. For t sufficiently large we obviously have that

$$\mathcal{L}^2(B, \kappa, Q) \;\subset\; \mathcal{A}_t\,, \tag{11.8}$$

where $\mathcal{A}_t$ is defined at the beginning of Section 8. Then, as a result of (8.8) and (8.27), we have that

$$\left|\mathcal{L}^2(B, \kappa, Q)\right| \;\leq\; \kappa\; |B| \tag{11.9}$$

provided that t is sufficiently large.

The desired estimate (11.1) now follows from (11.6), (11.7), and (11.9) with

$$C_0 := 3^n K_d + 1.$$

This thereby completes the proof of Theorem 1.5.

Funding

This work was supported by the Engineering and Physical Sciences Research Council Programme [grant EP/J018260/1 to V.B. and S. V.]; the United States-Israel Binational Science Foundation [grant 2016256 to A. Ganguly]; University Grants Commission, the Indo-French Centre for the Promotion of Advanced Research, and a Department of Science and Technology, Government of India Swarnajayanti fellowship [to A. Ghosh]; and the Benoziyo Endowment Fund for the Advancement of Science at the Weizmann Institute [to A. Ghosh].

Acknowledgments

We thank the referees for helpful suggestions. S. V. would like to thank the Tata Institute of Fundamental Research (Mumbai) for its hospitality during a productive and most enjoyable visit during March 2017. On a more personal note, many thanks to Anish and Dhru for sharing with me the wonderful culinary delights of Mumbai and the wonderful wedding party. Finally a great big thanks to the lovely Bridget and the feisty young women Iona and Ayesha for a wonderful 2017; especially the weeks during the summer in France!

References

[1] Aka, M., E. Breuillard, L. Rosenzweig, and N. de Saxce. “Diophantine approximation on matrices and lie groups.” *Geometric and Functional Analysis* 28, no. 1 (2018): 1–57.

[2] Badziahin, D., V. Beresnevich, and S. Velani. “Inhomogeneous theory of dual Diophantine approximation on manifolds.” *Adv. Math.* 232 (2013): 1–35.

[3] Beresnevich, V. “A Groshev type theorem for convergence on manifolds.” *Acta Math. Hungar.* 94, no. 1–2 (2002): 99–130.

[4] Beresnevich, V. and S. Velani. “An inhomogeneous transference principle and Diophantine approximation.” *Proc. Lond. Math. Soc.* 101 (2010): 821–51.

[5] Beresnevich, V. and S. Velani. “Simultaneous inhomogeneous Diophantine approximations on manifolds.” *Fundam. Prikl. Mat.* 16, no. 5 (2010): 3–17.

[6] Beresnevich, V., D. Dickinson, and S. Velani. “Measure theoretic laws for lim sup sets.” *Mem. Amer. Math. Soc.*, 179 (2006): pp. x+91.

[7] Beresnevich, V., F. Ramirez, and S. Velani. “Metric Diophantine Approximation: aspects of recent work.” In *Dynamics and Analytic Number Theory: Durham Easter School 2014 Proceedings*, edited by D. Badziahin, A. Gorodnik, and N. Peyerimhoff. LMS Lecture Note Series 437, United Kingdom: Cambridge University Press, (2016): 1–95.

[8] Beresnevich, V., V. Bernik, H. Dickinson, and M. M. Dodson. ”On linear manifolds for which the Khintchin approximation theorem holds.” *Vestsi Acad Navuk Belarusi. Ser. Fiz. - Mat. Navuk* 2 (2000): 14–7 (Belorussian).

[9] Beresnevich, V., V. Bernik, M. Dodson, and S. Velani. “Classical metric diophantine approximation revisited, Roth Festschrift - a special volume dedicated to Klaus Roth on his 80th birthday.” Edited by W. Chen, T. Gowers, H. Halberstam, W. M. Schmidt, and R. C. Vaughan. United Kingdom: Cambridge University Press, 2009. xviii+491 pp. ISBN: 978-0-521-51538-2.

[10] Beresnevich, V., V. Bernik, D. Kleinbock, and G. Margulis. “Metric Diophantine approximation: the Khintchine-Groshev theorem for non-degenerate manifolds.” *Moscow Math. J.* 2, no. 2 (2002): 203–25.

[11] Bernik, V., D. Kleinbock, and G. A. Margulis. “Khintchine type theorems on manifolds : the convergence case for the standard and multiplicative versions.” *Internat. Math. Res. Notices* 9 (2001): 453–86.

[12] Cassels, J. W. S. *An Introduction to Diophantine Approximation*. Cambridge: Cambridge University Press, 1957.

[13] Das, T., L. Fishman, D. Simmons, and M. Urbanski. "Extremality and dynamically defined measures, Part -I: Diophantine properties of quasi-decaying measures." *Selecta Math.* (2017): 1–42.

[14] Dickinson, H. and S. Velani. "Hausdorff measure and linear forms." *J. Reine Angew. Math.* 490 (1997): 1–36.

[15] Ganguly, A. and A. Ghosh. "Quantitative Diophantine approximation on affine subspaces." Math. Z. In press. https://arxiv.org/abs/1610.02157

[16] Ghosh, A. "A Khintchine-type theorem for hyperplanes." *J. London Math. Soc.* 72, no. 2 (2005): 293–304.

[17] Ghosh, A. "Diophantine approximation on affine hyperplanes." *Acta Arithmetica* 144 (2010): 167–82.

[18] Ghosh, A. "A Khintchine–Groshev theorem for affine hyperplanes." *Int. J. Number Theory* 7, no. 4 (2011): 1045–64.

[19] Ghosh, A. "Diophantine approximation and the Khintchine–Groshev theorem." *Monatsh. Math.* 163, no. 3 (2011): 281–99.

[20] Ghosh, A. "Diophantine approximation on subspaces of $\mathbb{R}^n$ and dynamics on homogeneous spaces." To appear in the Handbook of Group Actions III/IV, edited by L. Ji, A. Papadopoulos, and S. T. Yau.

[21] Ghosh, A. and A. Marnat. "On Diophantine transference principles." *Math. Proc. Cambridge Philos. Soc.*, https://arxiv.org/abs/1610.02161. https://arxiv.org/abs/1610.02161. https://doi.org/10.1017/S0305004118000014.

[22] Groshev, A. "Une théorème sur les systèmes des formes linéaires." *Dokl. Akad. Nauk SSSR* 9 (1938): 151–2.

[23] Khintchine, A. "Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen." *Math. Ann.* 92 (1924): 115–25.

[24] Kleinbock, D. "Extremal subspaces and their submanifolds." *Geom. Funct. Anal* 13, no. 2 (2003): 437–66.

[25] Kleinbock, D. "An extension of quantitative nondivergence and applications to Diophantine exponents." *Trans. Amer. Math. Soc.* 360, no. 12 (2008): 6497–523.

[26] Kleinbock, D. and G. A. Margulis. "Flows on homogeneous spaces and Diophantine approximation on manifolds." *Ann. of Math.* 148 (1998): 339–60.

[27] Sprindžuk, V. G. "Achievements and problems in Diophantine approximation theory." *Russian Math. Surveys* 35 (1980): 1–80.